Don't worry: just guess and you'll be fine (eventually)!

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Suppose that a sequence $(x_t)_{t=1}^{\infty} \in \{-1,1\}^{\infty}$ is given. Let $(Y_t)_{t=1}^{\infty}$ be a stochastic process in $\{-1,1\}^{\infty}$, and define the process $(x_t+Y_t)_{t=1}^{\infty}$ in $\{-2,0,2\}$.

Suppose that an individual is hired to forecast x_t at time $t-1$, and that only the realization of x_t+Y_t is observed at t (but neither x_t nor the realization of Y_t are directly observed). The individual is payed \$1 for his forecast, but if it is known at t that the forecast made at $t-1$ was wrong, then the individual is required to return the fee, and pay a penalty $\$\Delta$, with $\Delta \geq 0$. A forecast is detected to be wrong if the difference between it and the observation of x_t+Y_t is 3; otherwise, the difference is 1 and it is impossible to guarantee that a mistake was made: either the forecast is verified to be correct, or the random shock has offset the systemic effect and it is impossible to tell whether the forecast was right or wrong.

Now, suppose that the individual ignores the components of x, and, hence, decides to randomly forecast from $\{-1,1\}$ with equal probabilities. Denote by U_t the random variable representing the guesses.

I will show conditions under which, if there is no impatience or risk aversion, the individual will eventually (and permanently thereafter) benefit from his forecasts (make positive earnings).

Let π_t be the probability that $Y_t = 1$. And let $\mathbb{P}$ denote the probability distribution induced in $\{-1,1\}^{\infty} \times \{-1,1\}^{\infty}$ by the product of the uniform guesses and the $(\pi_t)_{t=1}^{\infty}$ distributions.

For every $T \in \mathbb{N}$, define $G_T : \{-1,1\}^{3T} \to \mathbb{R}$ by

$$G_T\left((u_t,x_t,y_t)_{t=1}^{T}\right) := \sum_{t=1}^{T} \left(I(u_t = x_t) + I(u_t = y_t \neq x_t) - \Delta I(x_t = y_t \neq u_t)\right),$$

and let $g_T := \frac{1}{T} G_T$

G_T measures the time-T cumulative earnings of the forecaster: along history $(u_t,x_t,y_t)_{t=1}^{T}$, the individual has kept \$1 whenever $|u_t - x_t - y_t| = 1$ and has

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paid, net of fee, \$$\Delta$ when $|u_t - x_t - y_t| = 3$. g_T measures his time-T average earnings.

Notice that if x_t were observed, then the time-T cumulative earnings of the forecaster would be simply $\sum_{t=1}^{T} (I(u_t = x_t) - \Delta I(x_t \neq u_t))$, or, equivalently, $\sum_{t=1}^{T} (I(u_t = x_t) - \Delta I(u_t = y_t \neq x_t) - \Delta I(x_t = y_t \neq u_t))$. *Then, the fact that only $x_t + Y_t$ is observed effectively changes the second term in favor of the forecaster.*

Thus, it is immediate that the worst-case scenario for the forecaster is when x_t takes the value that maximizes the probability that $Y_t = x_t$: given $\Delta \geq 1$,

$$x_t = -1 \Rightarrow \pi_t \leq 1 - \pi_t;$$
$$x_t = 1 \Rightarrow 1 - \pi_t \leq \pi_t;$$

As a limit case, suppose for the moment that

$$x_t = -1 \Rightarrow \pi_t = \tfrac{\Delta-1}{\Delta+1};$$

$$x_t = 1 \Rightarrow \pi_t = \tfrac{2}{\Delta+1};$$

and define the process $(Z_t)_{t=1}^{\infty}$ as follows:

$$Z_t := \begin{cases} 1, & \text{if } U_t = x_t; \\ 1, & \text{if } U_t = Y_t \neq x_t; \\ -\Delta, & \text{otherwise.} \end{cases}$$

Let P_t denote the distribution of Z_t. That $(Z_t)_{t=1}^{\infty}$ is independently distributed follows by assumption. Now, to see that it is identically distributed as well, notice that if $x_t = -1$, then

$$P_t(Z_t = 1) = \frac{1}{2} + \frac{1}{2}\frac{\Delta - 1}{\Delta + 1},$$

whereas if $x_t = 1$, then

$$P_t(Z_t = 1) = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{2}{\Delta + 1}\right) = \frac{1}{2} + \frac{1}{2}\frac{\Delta - 1}{\Delta + 1}.$$

Also, notice that

$$\mathbf{E}\left[Z_t\right] = \frac{1}{2} + \frac{1}{2}\frac{\Delta - 1}{\Delta + 1} - \Delta\left(\frac{1}{2} - \frac{1}{2}\frac{\Delta - 1}{\Delta + 1}\right) = 0,$$

so it follows from the Strong Law of Large Numbers that

$$\frac{1}{T}\sum_{t=1}^{T} Z_t \xrightarrow{a.s.} 0,$$

as $T \to \infty$.

Now notice that Z_t measures period-T earnings under the worst-case scenario for the forecaster. So, in this worst-case scenario his average earnings converge to zero with probability 1.

It follows that if there exist $\underline{\pi} > \frac{\Delta-1}{\Delta+1}$ and $\overline{\pi} < \frac{2}{\Delta+1}$ such that for every $t \in \mathbb{N}$, $\pi_t \in [\underline{\pi}, \overline{\pi}]$, then, even in the worst case, the forecaster makes positive average (and hence aggregate) earnings in the long run with probability 1.

For example, suppose that the deviations of the exchange rate from its equilibrium are determined by the sum of a systemic term, x, and a shock Y. If both terms move in the same direction, either depreciation, $x = Y = -1$, or appreciation, $x = Y = 1$, then the exchange rate moves in that direction. Otherwise, the effects cancel each other out and the exchange rate remains in equilibrium. An economist is hired as consultant to forecast the systemic term. The economist is clueless about the systemic effects, so he just tosses a coin. Even if his detected mistakes are penalized by more than the fees he receives for his arduous work, if the distribution of random shocks is bounded sufficiently away from degeneracy, we have shown that, just by guessing, the economist will make positive profits, almost surely, in the long run.